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***Choosability of the square of planar subcubic graphs  
with large girth***

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## Choosability of the square of planar subcubic graphs with large girth

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**Abstract:** We first show that the choose number of the square of a subcubic graph with maximum average degree less than  $18/7$  is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

**Key-words:** colouring, list colouring, planar graph, girth

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## Choisissabilité du carré des graphes planaires subcubiques de grande maille

**Résumé :** Nous montrons que le nombre de choix du carré d'un graphe subcubique de degré moyen maximum inférieur à  $18/7$  est au plus 6. Ceci implique que le carré d'un graphe planaire subcubique de maille au moins 9 est 6-choisissable. Ensuite, nous montrons que le nombre de choix du carré d'un graphe planaire subcubique de maille au moins 13 est au plus 5.

**Mots-clés :** coloration, coloration par listes, graphe planaire, maille

# 1 Introduction

Let  $G$  be a (simple) graph.

Let  $v$  be a vertex of  $G$ . Its *neighbourhood*, denoted  $N_G(v)$  is the set of its *neighbours*, i.e. is vertices  $y$  such that  $xy$  is an edge. The *degree* of a vertex  $v$  in  $G$ , denoted  $d_G(v)$ , is its number of neighbours. Often, when the graph  $G$  is clearly understood from the context, we omit the subscript  $G$ .

A graph is *subcubic* if every vertex has degree at most 3.

Let  $p : V(G) \rightarrow \mathbb{N}$ . A *p-list-assignment* is a list-assignment  $L$  such that  $L(v) = p(v)$  for any  $v \in V(G)$ .  $G$  is *p-choosable* if it is  $L$ -colourable for any  $p$ -list-assignment. By extension, if  $k$  is an integer, we say that  $G$  is *k-choosable* if it is  $p$ -choosable when  $p$  is the constant function with value  $k$  (i. e.  $p(v) = k$  for all  $v \in V$ ). The *choose number* of  $G$  is the smallest integer  $k$  such that  $G$  is  $k$ -choosable. Clearly the choose number of  $G$  is at least as large as  $\chi(G)$  the *chromatic number* of  $G$ .

The *square* of  $G$ , is the graph  $G^2$  with vertex set  $V(G)$  such that two vertices are linked by an edge of  $G^2$  if and only if  $x$  and  $y$  are at distance at most 2 in  $G$  (either  $xy$  is an edge or  $x$  and  $y$  share a neighbour). Formally,  $E(G^2) = \{xy, (xy \in E(G) \text{ or } \exists z \in N(x) \cap N(y))\}$ .

Wegner [4] proved that the square of a subcubic planar is 8-colourable. He also conjectured it is 7-colourable.

**Conjecture 1 (Wegner [4])** *Let  $G$  be a subcubic planar graph. Then  $\chi(G^2) \leq 7$ .*

The *average degree* of  $G$ , denoted  $Ad(G)$  is  $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$ .

The *maximum average degree* of  $G$ , denoted  $Mad(G)$ , is  $\max\{Ad(H), H \text{ subgraph of } G\}$ .

In [1], Dvořák, Škrekovski and Tancer proved that the choose number of the square of a subcubic graph  $G$  is at most 4 if  $Mad(G) < 24/11$  and  $G$  has no 5-cycle, at most 5 if  $Mad(G) < 7/3$  and at most 6 if  $Mad(G) < 5/2$ .

A graph is called *planar* if it can be embedded in the plane. The *girth* of a graph is the smallest length of a cycle in  $G$ . Planar graph with prescribed girth have bounded maximum average degree:

**Proposition 2** *Every planar graph with girth at least  $g$  has maximum average degree less than  $2 + \frac{4}{g-2}$ .*

Hence the results of Dvořák, Škrekovski and Tancer imply that the choose number of the square of a planar graph with girth  $g$  is at most 6 if  $g \geq 13$ , at most 5 if  $g \geq 14$  and at most 4 if  $g \geq 24$ . The two later results had been previously proved by Montassier and Raspaud [3].

In this paper, we improve some of these results. We first show (Theorem 4) that the choose number of the square of a subcubic graph with maximum average degree less than  $18/7$  is at most 6. As a corollary, we get that the choose number of the square of a planar graph with girth at least 9 is at most 6. We then show (Theorem 9) that the choose number of the square of a subcubic planar graph with girth at least 13 is at most 5.

## 2 The main results

The general frame of the proofs are classical. We consider a  $k$ -minimal graph, that is a subcubic graph such that its square is not  $k$ -choosable, but the square of every proper subgraph is  $k$ -choosable. We prove that some *configurations* (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction.

An  $i$ -vertex is a vertex of degree  $i$ . We denote by  $V_i$  the set of  $i$ -vertices of  $G$  and by  $v_i$  its cardinality. Let  $v$  be a vertex. An  $i$ -neighbour of  $v$  is a neighbour of  $v$  with degree  $i$ . The  $i$ -neighbourhood of  $v$  is  $N_i(v) = N(v) \cap V_i$  and its  $i$ -degree is  $d_i(v) = |N_i(v)|$ .

Some properties of 6- and 5-minimal graphs have already been proved in [1]. The easy first one is that  $V_0 \cup V_1 = \emptyset$ , so  $G$  has minimum degree 2. This will allow us to use the following definitions for 6- and 5-minimal graphs.

**Definition 3** Let  $G$  be a subcubic graph with minimum degree 2.

A *thread* of  $G$  is a maximal induced path of  $G$ . In other words, it is a path whose endvertices are 3-vertices and whose internal vertices are 2-vertices.

The *kernel* of  $G$  is the weighted graph  $K$  such that  $V(K) = V_3(G)$  and  $xy$  is an edge in  $K$  with weight  $l$  if and only if  $x$  and  $y$  are connected by a thread of length  $l$  in  $G$ . An edge of weight  $l$  is also called  $l$ -edge.

Let  $x$  be a 3-vertex of  $G$ . The *type* of  $x$  is the triple  $(l_1, l_2, l_3)$  such that  $l_1 \leq l_2 \leq l_3$  and the three edges (A loop is counted twice.) incident to  $x$  have weight  $l_1$ ,  $l_2$  and  $l_3$  in  $K$ . We denote by  $Y_{l_1, l_2, l_3}$  the set of 3-vertices of type  $(l_1, l_2, l_3)$  and  $y_{l_1, l_2, l_3}$  its cardinality. Moreover, for every integer  $i$ , we define  $Z_i := \bigcup_{l_1+l_2+l_3=i} Y_{l_1, l_2, l_3}$  and  $z_i = |Z_i|$ .

The number of vertices and edges and thus the average degree of a subcubic graph  $G$  with minimum degree 2 may be easily expressed in terms of the  $z_i$ :

$$\begin{aligned} |V(G)| &= \sum_{i \geq 3} \frac{i-1}{2} z_i \\ 2|E(G)| &= \sum_{i \geq 3} i \cdot z_i \\ Ad(G) &= \frac{\sum_{i \geq 3} i \cdot z_i}{\sum_{i \geq 3} \frac{i-1}{2} z_i} \end{aligned} \tag{1}$$

### 2.1 6-choosability

The aim of this subsection is to prove the following result.

**Theorem 4** Let  $G$  be a subcubic graph of maximum average degree  $d < 18/7$ . Then  $G^2$  is 6-choosable.

In order to prove this theorem, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [1].

**Lemma 5 (Dvořák, Škrekovski and Tancer [1])** *Let  $G$  be a 6-minimal graph. Then the following hold:*

- 1) *all the edges of  $K$  have weight at most 2;*
- 2) *every 3-cycle of  $G$  has its vertices in  $V_3$ ;*
- 3) *every 4-cycle of  $G$  has at least three vertices in  $V_3$ ;*
- 4) *a vertex of  $Y_{2,2,2}$  is not adjacent to a vertex of  $Y_{1,2,2} \cup Y_{2,2,2}$ .*

We will prove in Subsection 3.2 some new properties.

**Lemma 6** *Let  $G$  be a 6-minimal graph. Then the following hold:*

- 5) *if  $(v_1, v_2, v_3, v_4, v_1)$  is a 4-cycle with  $v_2 \in V_2$  then  $v_1$  or  $v_3$  is not in  $Z_5$ ;*
- 6) *a vertex of  $Y_{1,2,2}$  is adjacent to at most one vertex of  $Y_{1,2,2}$  by 2-edges.*

**Proof of Theorem 4.** Let  $G$  be a 6-minimal planar graph.  $G$  has minimum degree 2, so its kernel  $K$  is defined. Moreover by Lemma 5-1),  $Z_i$  is empty for  $i \geq 7$  and  $Z_6 = Y_{2,2,2}$  and  $W_5 = Y_{1,2,2}$ .

Let us consider a vertex of  $Z_4 = Y_{1,1,2}$ . Its neighbour via the 2-edge is in  $Z_4 \cup Z_5 \cup Z_6$  because a vertex of  $Z_3 = Y_{1,1,1}$  is incident to no edge of weight 2.

For  $i = 4, 5, 6$ , let  $Z_4^i$  be the set of vertices of  $Z_4$  which are incident to a vertex of  $Z_i$  by their unique 2-edge and  $z_4^i$  its cardinality.  $(Z_4^4, Z_4^5, Z_4^6)$  is a partition of  $Z_4$  so  $z_4 = z_4^4 + z_4^5 + z_4^6$ .

Hence Equation (1) becomes

$$Ad(G) = \frac{6z_6 + 5z_5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{\frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4^6 + \frac{5}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma 5-4), the three neighbours of a vertex of  $Z_6$  is not in  $Z_6 \cup Z_5$ . So they must be in  $Z_4^6$ . It follows that  $3z_6 = z_4^6$ . So

$$Ad(G) = \frac{5z_5 + 6z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{2z_5 + \frac{7}{3}z_4^6 + \frac{5}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

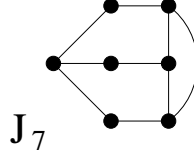
By Lemma 6-6), a vertex of  $Z_5$  is adjacent to at least one vertex of  $Z_4^5$ . Thus  $z_5 \leq z_4^5$ .

But  $Ad(G)$  is decreasing as a function of  $z_5$  since  $z_4^6, z_4^5, z_4^4$  and  $z_3$  are non-negative. It follows that

$$Ad(G) \geq \frac{6z_4^6 + 9z_4^5 + 4z_4^4 + 3z_3}{\frac{7}{3}z_4^6 + \frac{7}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3} \geq \frac{18}{7}.$$

□



Figure 1: The graph  $J_7$ 

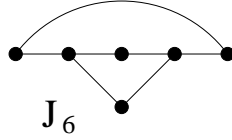
**Remark 7** Theorem 4 is tight. Indeed, the graph  $J_7$  depicted Figure 1 has average degree  $18/7$  and its square is the complete graph on seven vertices  $K_7$  which is not 6-choosable (nor 6-colourable).

Theorem 4 and Proposition 2 yield that the square of a planar graph with girth 9 is 6-choosable.

**Corollary 8** *The square of a planar graph with girth 9 is 6-choosable.*

## 2.2 5-choosability

Dvořák, Škrekovski and Tancer [1] proved that the square of a subcubic graph  $G$  with maximum average degree less than  $7/3$  is 5-choosable. This result is tight since the graph  $J_6$  depicted Figure 2 has average degree  $7/3$  and its square is the complete graph on six vertices  $K_6$  which is not 5-choosable (nor 5-colourable).

Figure 2: The graph  $J_6$ 

However, we will prove that the square of a planar graph with girth at least 13 is 5-choosable, which improves the result of Montassier and Raspaud.

**Theorem 9** *The square of a planar graph with girth 13 is 5-choosable.*

In order to prove this theorem, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [1].

**Lemma 10 (Dvořák, Škrekovski and Tancer [1])** *Let  $G$  be a 5-minimal graph of girth 13. Then the following hold:*

- 1) all the edges of  $K$  have weight at most 3;
- 2) if  $i \geq 8$ ,  $Z_8$  is empty.

We will prove in Subsection 3.3 some new properties.

**Lemma 11** *Let  $K$  be the  $G$  be a 5-minimal graph of girth at least 13. Then the following holds:*

- 3) A vertex of  $Y_{2,2,3}$  and a vertex of  $Y_{1,2,3} \cup Y_{2,2,3}$  are not linked by a 2-edge.
- 4) A vertex of  $Y_{1,3,3}$  and a vertex of  $Y_{1,2,3} \cup Y_{1,3,3}$  are not linked by a 1-edge.
- 5) A vertex of  $Y_{2,2,2}$  is not adjacent to three vertices of  $Y_{2,2,3}$  (by 2-edges).

**Proof of Theorem 9.** Let  $G$  be a 5-minimal planar graph with girth at least 13.  $G$  has minimum degree 2, so its kernel  $K$  is defined. Moreover, by Lemma 10-1),  $Z_7 = Y_{2,2,3} \cup Y_{1,3,3}$ , so

$$z_7 = y_{2,2,3} + y_{1,3,3}. \quad (2)$$

Let us count the number  $e_2$  of 2-edges incident to vertices of  $Y_{2,2,3}$ . Since 2-edges may not link two vertices of type  $(2, 2, 3)$  according to Lemma 11-3), we have  $e_2 = 2y_{2,2,3}$ . Moreover, the end of such edges which is not in  $Y_{2,2,3}$  has to be in  $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_4$  by Lemmas 10 and 11-3). Furthermore, a vertex of  $Y_{2,2,2}$  is incident to at most two edges of  $e_2$  according to Lemma 11-5) and a vertex of  $Y_{1,2,2}$  (resp.  $Z_4$ ) is incident to at most two (resp. one) 2-edges. Therefore  $e_2 \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4$ . So,

$$2y_{2,2,3} \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4 \quad (3)$$

Let us now count the number  $e_1$  of 1-edges incident to vertices of  $Y_{1,3,3}$ . Since 1-edges may not link two vertices of type  $(1, 3, 3)$  according to Lemma 11-4), we have  $e_1 = y_{1,3,3}$ . Moreover, the end of such edges which is not in  $Y_{2,2,3}$  has to be in  $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_4 \cup Z_3$  by Lemmas 10 and 11-4). Furthermore, vertices of  $Y_{1,2,2}$  (resp.  $Y_{1,1,3} \cup Z_4, Z_3$ ) are incident to at most one (resp. two, three) 1-edges. Thus  $e_1 \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3$ . So,

$$y_{1,3,3} \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3 \quad (4)$$

$2 \times (4) + (3)$  yields  $2y_{2,2,3} + 2y_{1,3,3} \leq 2y_{2,2,2} + 4y_{1,2,2} + 4y_{1,1,3} + 5z_4 + 6z_3$ . Hence by Equation 2,  $2z_7 \leq 2z_6 + 4z_5 + 5z_4 + 6z_3$ , so

$$z_7 \leq z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3.$$

Now by Equation 1 the average degree of  $G$  is

$$Ad(G) = \frac{7z_7 + 6z_6 + 5z_5 + 4z_4 + 3z_3}{3z_7 + \frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4 + z_3}.$$

As a function of  $z_7$  this is a decreasing function (on  $\mathbb{R}^+$ ), so it is minimum when  $z_7$  is maximum that is equal to  $z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3$ . So, :

$$Ad(G) \geq \frac{13z_6 + 19z_5 + \frac{43}{2}z_4 + 24z_3}{\frac{11}{2}z_6 + 8z_5 + 9z_4 + 10z_3} \geq \frac{26}{11}.$$

This contradicts the fact that  $G$  has girth 13 by Proposition 2.  $\square$

It is very likely that using the method below, one can prove that a graph  $G$  with maximum average degree less than  $\frac{26}{11}$  is 5-choosable unless it contains  $J_6$  as an induced subgraph. However, this will require the tedious study of a large number of configurations.

### 3 Proofs of Lemmas 6 and 11

In order, to prove Lemmas 6 and 11, we need the following lemma proved in [1]. Let  $S$  be a set of vertices of a  $k$ -minimal graph  $G$ . The function  $p_S : S \rightarrow \mathbb{N}$  is defined by  $p_S(v) = k - |N_{G^2}(v) \setminus S|$ . Then  $p_S(v)$  represents the minimum number of available colours at a vertex  $v \in S$  once we have precoloured the square of  $G - S$ . Hence if  $(G - S)^2$  is 6-choosable,  $(G - S)^2 = G^2 - S$  (in particular, it happens when two distinct vertices of  $G - S$  have no common neighbour in  $S$ ) and  $G[S]^2$  is  $p_S$ -choosable, one can extend any  $k$ -list-colouring of  $G - H$  into a  $k$ -list-colouring of  $G$ .

**Lemma 12 (Dvořák, Škrekovski and Tancer [1])** *Let  $S$  be a set of vertices of a  $k$ -minimal graph  $G$ . If  $(G - S)^2 = G^2 - S$ , then  $G^2[S]$  is not  $p_S$ -choosable.*

In order to use Lemma 12, we need some results on the choosability of some graphs.

#### 3.1 Some choosability tools

**Definition 13** Let  $x$  and  $y$  be two vertices of a graph  $G$ . An  $(x - y)$ -ordering of  $G$  is an ordering such that  $x$  is the minimum and  $y$  the maximum. An  $(x, y - z)$ -ordering is an ordering such that  $x$  is minimum,  $y$  is the second minimum and  $z$  is maximum.

Let  $\sigma = (v_1 < v_2, \dots < v_n)$  be an ordering of the vertices of  $G$ .  $\sigma$  is  $p$ -greedy if for every  $i$ ,  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| < p(v_i)$ . It is  $p$ -nice if for every  $i$  except  $n$ ,  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| < p(v_i)$  and  $d(v_n) = p(v_n)$ . It is  $p$ -good if for every  $3 \leq i \leq n$ ,  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| - \epsilon(v_i) < p(v_i)$  with  $\epsilon(v_i) = 1$  if  $v_i$  is adjacent to both  $v_1$  and  $v_2$  and  $\epsilon(v_i) = 0$  otherwise.

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.

**Lemma 14** *If  $G$  has a  $p$ -greedy ordering then  $G$  is  $p$ -choosable.*

**Proof.** Applying the greedy algorithm according to the  $p$ -greedy ordering gives the desired colouring.  $\square$

**Lemma 15** *Let  $xy$  be an edge of graph  $G$  and  $L$  be a  $p$ -list-assignment of  $G$ . If  $L(x) \not\subset L(y)$  and  $G$  has a  $p$ -nice  $(x - y)$ -ordering, then  $G$  is  $L$ -colourable.*

**Proof.** Let  $a$  be a colour in  $L(x) \setminus L(y)$ . Proceed the greedy algorithm starting by assigning  $a$  to  $x$ . The only vertex which has not more colour in its list than previously coloured neighbours is  $y$  for which  $|L(y)| = d(y)$ . But since  $a \notin L(y)$ , at most  $d(y) - 1$  colours of  $L(y)$  are assigned to the neighbours of  $y$ . Hence one can colour  $y$ .  $\square$

**Lemma 16** *Let  $x, y$  and  $z$  be three vertices of a graph  $G = (V, E)$  such that  $xy \notin E$ ,  $xz, yz \in E$ . If  $L(x) \cap L(y) \neq \emptyset$  and  $G$  has a  $p$ -good  $(x, y - z)$ -ordering, then  $G$  is  $L$ -colourable.*

**Proof.** Let  $a$  be a colour in  $L(x) \cap L(y)$  and  $\sigma = (v_1 < v_2, \dots < v_n)$  be a  $p$ -good  $(x, y - z)$ -ordering. (In particular,  $v_1 = x, v_2 = y$  and  $v_n = z$ .) Proceed the greedy algorithm according to  $\sigma$  starting by assigning  $a$  to  $x$  and  $y$ . For every  $3 \leq i \leq n$ , the number of colours assigned to already coloured neighbours of  $v_i$  is at most  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| - \epsilon(v_i)$  since  $v_1$  and  $v_2$  are coloured the same. Hence the greedy algorithm gives an  $L$ -colouring.  $\square$

**Remark 17** Note that under the condition  $xz, yz \in E$ , a  $p$ -nice ordering is also a  $p$ -good ordering.

**Definition 18** The *blocks* of a graph are its maximal 2-connected components. A connected graph is said to be a *Gallai tree* if each of its blocks are either complete graphs or odd cycles.

**Theorem 19 (trouver si c'est Lovasz, Borodin, erdos et al)** *Let  $G$  be a connected graph. Then  $G$  is  $d$ -choosable if and only if  $G$  is not a Gallai tree.*

**Lemma 20** *Let  $G = (V, E)$  be a graph and  $p : V(G) \rightarrow \mathbb{N}$ . Let  $S$  be a set of vertices such that  $p(v) \geq d(v)$  for all  $v \in S$ . If  $G[S]$  is not a Gallai tree and  $G - S$  is  $p$ -choosable then  $G$  is  $p$ -choosable.*

**Proof.** Let  $L$  be a  $p$ -list-assignment of  $G$ . Since  $G - S$  is  $p$ -choosable, it admits a  $p$ -colouring  $c$ . Let us now extend it to  $S$ . The list  $I(v) = L(v) \setminus \{c(w), w \in N(v) \setminus S\}$  of available colours of a vertex  $v \in S$  is of size at least  $p'(v) = p(v) - |N(v) \setminus S| \geq d_{G[S]}(v)$ . Since  $G[S]$  is not a Gallai tree, by Theorem 19,  $G[S]$  is  $p'$ -choosable and thus  $I$ -colourable. So,  $G$  is  $L$ -colourable.  $\square$

A 4-regular graph is *cycle+triangles* if it is the edge union of a Hamiltonian cycle and triangles.

**Theorem 21 (Fleishner and Stiebitz [2])** *Every cycle+triangles graph is 3-choosable.*

### 3.2 Proof of Lemma 6

**Lemma 22** *Let  $q \geq 4$  and  $C_{2q} = (v_1, \dots, v_{2q}, v_1)$  be the  $2q$ -cycle and  $p$  defined by  $p(v_i) = 4$  if  $i$  is odd and  $p(v_i) = 3$  otherwise. Then  $C_{2q}^2$  is  $p$ -choosable.*

**Proof.** The set  $S$  of vertices  $v$  for which  $p(v) \geq d_{C_{2q}^2}(v)$  is the set of  $v_i$  with odd indices.  $C_{2q}^2[S]$  is a  $q$ -cycle and thus is not a Gallai tree if  $q \geq 4$ . Moreover  $C_{2q}^2 - S$  is also a  $q$ -cycle and is 3-choosable. Hence Lemma 20 gives the result.  $\square$

**Proposition 23** *Let  $P_7 = (v_1, \dots, v_7)$  be a path and  $p$  the function defined by  $p(v_1) = p(v_2) = p(v_6) = p(v_7) = 2$ ,  $p(v_3) = p(v_5) = 4$  and  $p(v_4) = 3$ . Then  $P_7^2$  is  $p$ -choosable.*

**Proof.** Since  $(v_2 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$  is  $p$ -nice, by Lemma 15, we may assume that  $L(v_1) = L(v_2)$ , and by symmetry of  $P_7$  and  $p$  that  $L(v_6) = L(v_7)$ .

Since  $(v_1 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$  is  $p$ -good, by Lemma 16, we may assume that  $L(v_1) \cap L(v_4) = \emptyset$ , and by symmetry  $L(v_7) \cap L(v_4) = \emptyset$ .

Now one can find  $c(v_1) \in L(v_1)$ ,  $c(v_2) \in L(v_2) \setminus \{c(v_1)\}$ ,  $c(v_6) \in L(v_6)$ ,  $c(v_7) \in L(v_7) \setminus \{c(v_6)\}$ ,  $c(v_3) \in L(v_3) \setminus \{c(v_1), c(v_2)\}$ , and  $c(v_5) \in L(v_5) \setminus \{c(v_3), c(v_6), c(v_7)\}$ . Now since  $L(v_1) \cap L(v_4) = \emptyset$  and  $L(v_1) = L(v_2)$ ,  $c(v_2) \notin L(v_4)$ . Analogously,  $c(v_6) \notin L(v_4)$ . Hence,  $= L(v_4) \setminus \{c(v_2), c(v_3), c(v_5), c(v_6)\} = L(v_4) \setminus \{c(v_3), c(v_5)\} \neq \emptyset$ . So, one can choose  $c(v_4)$  in this set to get an  $L$ -colouring  $c$  of  $P_7^2$ .  $\square$

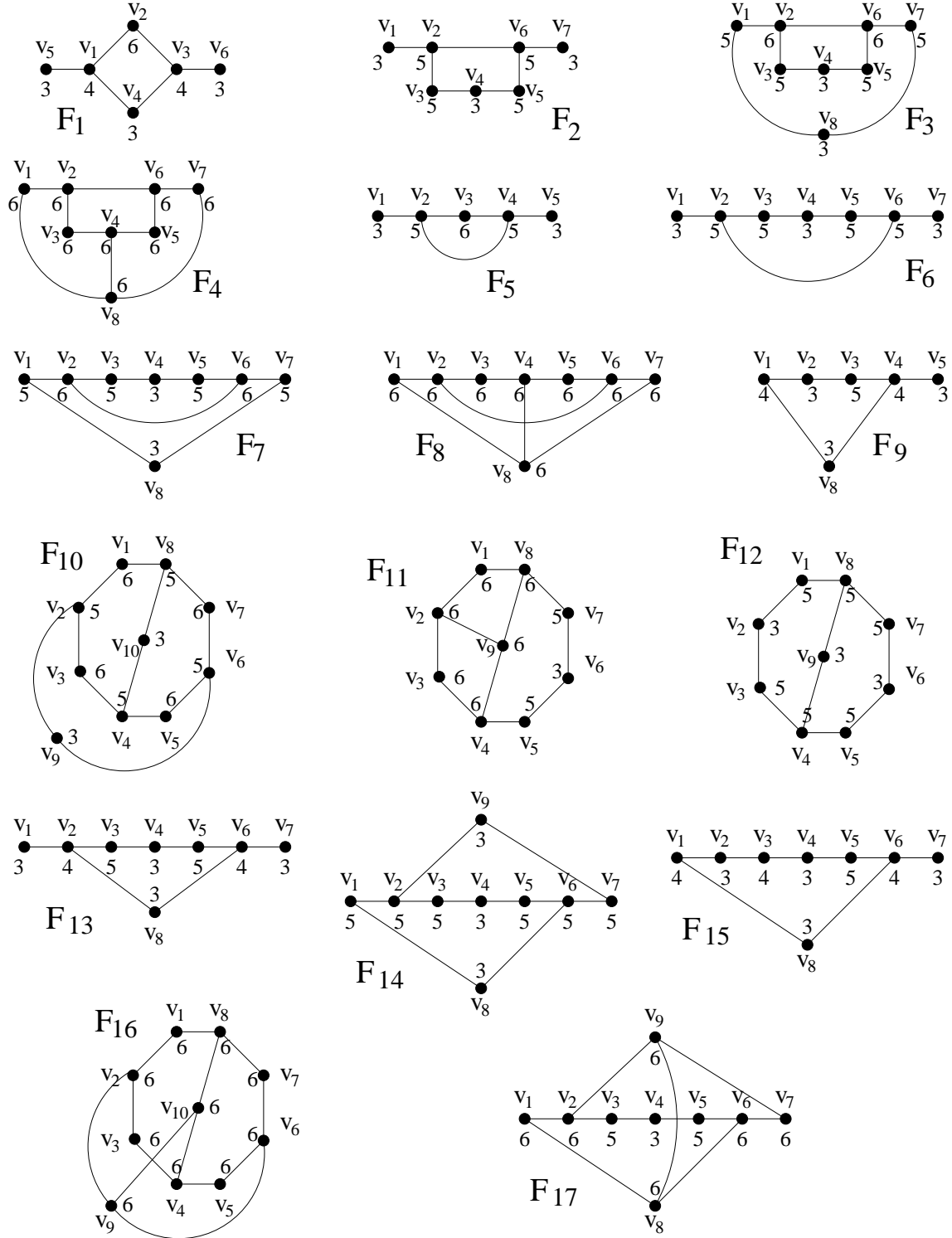
**Lemma 24** *For  $1 \leq i \leq 17$ , let  $F_i$  be the graphs and  $p_i$  be the function depicted Figure 3.*

- a)  $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$  is  $p_1$ -choosable.
- b)  $F_2^2 \cup \{v_1v_4\}$  and  $F_2^2 \cup \{v_4v_7\}$  are  $p_2$ -choosable.
- c)  $F_3^2 \cup \{v_4v_8\}$  is  $p_3$ -choosable.
- d)  $F_4^2$  is 6-choosable.
- e)  $F_5^2 \cup \{v_1v_5\}$  is  $p_5$ -choosable.
- f)  $F_6^2 \cup \{v_1v_4\}$  and  $F_6^2 \cup \{v_4v_7\}$  are  $p_6$ -choosable.
- g)  $F_7^2 \cup \{v_4v_8\}$  is  $p_7$ -choosable.
- h)  $F_8^2$  is 6-choosable.
- i)  $F_9^2$  is  $p_9$ -choosable.
- j)  $F_{10}^2 \cup \{v_9v_{10}\}$  is  $p_{10}$ -choosable.
- k)  $F_{11}^2$  is  $p_{11}$ -choosable.

- l)  $F_{12}^2 \cup \{v_2v_9\}$  and  $F_{12}^2 \cup \{v_6v_9\}$  are  $p_{12}$ -choosable.
- m)  $F_{13}^2 \cup \{v_4v_8\}$  is  $p_{13}$ -choosable.
- n)  $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$  is  $p_{14}$ -choosable.
- o)  $F_{15}^2 \cup \{v_4v_8\}$  is  $p_{15}$ -choosable.
- p)  $F_{16}^2$  is 6-choosable.
- q)  $F_{17}^2$  is  $p_{17}$ -choosable.

**Proof.**

- a) In  $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$ ,  $(v_6 < v_5 < v_4 < v_3 < v_1 < v_2)$  is  $p_1$ -greedy. So by Lemma 14,  $F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$  is  $p_1$ -choosable.
- b) In  $F_2^2 \cup \{v_4v_7\}$ ,  $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$  is  $p_2$ -nice and  $p_2(v_2) > p_2(v_1)$ . So by Lemma 15,  $F_2^2 \cup \{v_4v_7\}$  is  $p_2$ -choosable.  
By symmetry, one shows that  $F_2^2 \cup \{v_1v_4\}$  is  $p_2$ -choosable.
- c) In  $F_3^2 \cup \{v_4v_8\}$ ,  $(v_2 < v_8 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$  is  $p_3$ -nice and  $p_3(v_2) > p_3(v_1)$ . So by Lemma 15,  $F_3^2 \cup \{v_4v_8\}$  is  $p_3$ -choosable.
- d) Let  $L$  be a 6-list-assignment of  $F_4^2$ .  $(v_3 < v_2 < v_4 < v_6 < v_8 < v_5 < v_7 < v_1)$  is 6-nice. Thus, by Lemma 15, we may assume that  $L(v_3) = L(v_1)$ . Now,  $(v_5 < v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_3)$  is 6-nice. Hence  $F_4^2$  is  $L$ -colourable according to Lemma 16 if  $L(v_5) \cap L(v_1) \neq \emptyset$  or Lemma 15 otherwise.
- e) In  $F_5^2 \cup \{v_1v_5\}$ ,  $(v_1 < v_5 < v_2 < v_4 < v_3)$  is  $p_5$ -greedy. So by Lemma 14,  $F_5^2 \cup \{v_1v_5\}$  is  $p_5$ -choosable.
- f) In  $F_6^2 \cup \{v_4v_7\}$ ,  $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$  is  $p_6$ -nice and  $p_6(v_2) > p_6(v_1)$ . So by Lemma 15,  $F_6^2 \cup \{v_4v_7\}$  is  $p_6$ -choosable.  
By symmetry, one shows that  $F_6^2 \cup \{v_1v_4\}$  is  $p_6$ -choosable.
- g) In  $F_7^2 \cup \{v_4v_8\}$ ,  $(v_2 < v_4 < v_8 < v_5 < v_7 < v_6 < v_3 < v_1)$  is  $p_7$ -nice and  $p_7(v_2) > p_7(v_1)$ . So by Lemma 15,  $F_7^2 \cup \{v_4v_8\}$  is  $p_7$ -choosable.
- h) Let  $L$  be a  $p_8$ -list-assignment of  $F_8^2$ ,  $(v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_5 < v_3)$  is  $p_8$ -greedy, so by Lemma 15, we may assume that  $L(v_1) = L(v_3)$ . Now  $(v_3 < v_1 < v_2 < v_4 < v_6 < v_8 < v_7 < v_5)$ , so by Lemma 16,  $F_8^2$  is  $L$ -colourable.
- i) In  $F_9^2$ ,  $(v_4 < v_2 < v_8 < v_1 < v_3 < v_5)$  is  $p_9$ -nice and  $p_9(v_4) > p_9(v_5)$ . So by Lemma 15,  $F_9^2$  is  $p_9$ -choosable.



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Figure 3: The graphs  $F_i$  and functions  $p_i$  for  $1 \leq i \leq 12$

- j) Let  $L$  be a  $p_{10}$ -list-assignment of  $F_{10}^2 \cup \{v_9v_{10}\}$ . ( $v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1$ ) is  $p_{10}$ -nice. Thus, by Lemma 15, we may assume that  $L(v_2) \subset L(v_1)$ . Analogously, by symmetry, we may assume that  $L(v_2) \subset L(v_3)$  and  $L(v_4) \subset L(v_3)$ . It follows that  $L(v_1) \cap L(v_4) \neq \emptyset$ . Because  $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$  is  $p_{10}$ -good, by Lemma 16,  $F_{10}^2 \cup \{v_9v_{10}\}$  is  $L$ -colourable.
  - k) In  $F_{11}^2$ ,  $(v_6 < v_5 < v_7 < v_9 < v_8 < v_4 < v_3 < v_2 < v_1)$  is  $p_{11}$ -greedy. So by Lemma 14,  $F_{11}^2$  is  $p_{11}$ -choosable.
  - l) Let  $L$  be a  $p_{12}$ -list-assignment of  $F_{12}^2 \cup \{v_2v_9\}$ . Then  $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_3 < v_1)$  and  $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_1 < v_3)$  are  $p_{12}$ -nice so by Lemma 15, we may assume that  $L(v_2) \subset L(v_3) \cap L(v_1)$ . Moreover,  $(v_4 < v_2 < v_9 < v_6 < v_8 < v_7 < v_5 < v_1 < v_3)$  is  $p_{12}$ -nice so by Lemma 15, we may assume that  $L(v_4) \subset L(v_3)$ . It follows that  $L(v_1) \cap L(v_4) \neq \emptyset$ . Thus, by Lemma 16, since  $(v_1 < v_4 < v_2 < v_9 < v_8 < v_6 < v_7 < v_5 < v_3)$  is  $p_{12}$ -good,  $F_{12}^2 \cup \{v_2v_9\}$  is  $L$ -colourable.
- By symmetry, one shows that  $F_{12}^2 \cup \{v_6v_9\}$  is  $p_{12}$ -choosable.
- m) In  $F_{13}^2 \cup \{v_4v_8\}$ ,  $(v_2 < v_8 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$  is  $p_{13}$ -nice and  $p_{13}(v_2) > p_{13}(v_1)$ . So by Lemma 15,  $F_{13}^2 \cup \{v_4v_8\}$  is  $p_{13}$ -colourable.
  - n) Let  $L$  be a  $p_{14}$ -list-assignment of  $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$  ( $v_1 < v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3$ ) is  $p_{14}$ -nice. So by Lemma 15,  $L(v_1) = L(v_3)$ . Now,  $(v_4 < v_1 < v_8 < v_9 < v_2 < v_6 < v_7 < v_5 < v_3)$  is  $p_{14}$ -nice. Hence  $F_{14}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$  is  $L$ -colourable according to Lemma 16 if  $L(v_4) \cap L(v_1) \neq \emptyset$  or Lemma 15 otherwise.
  - o) In  $F_{15}^2 \cup \{v_4v_8\}$ .  $(v_6 < v_8 < v_4 < v_2 < v_1 < v_3 < v_5 < v_7)$  is  $p_{15}$ -nice and  $p_{15}(v_6) > p_{15}(v_7)$ . So by Lemma 15,  $F_{15}^2 \cup \{v_4v_8\}$  is  $p_{15}$ -choosable.
  - p) Let  $L$  be a 6-list-assignment of  $F_{16}^2$ . ( $v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1$ ) is 6-nice. Thus, by Lemma 15, we may assume that  $L(v_2) = L(v_1)$ . Analogously, by symmetry, we may assume that  $L(v_2) = L(v_3) = L(v_4)$ . It follows that  $L(v_1) = L(v_4)$ . Because  $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$  is 6-good, by Lemma 16,  $F_{16}^2$  is  $L$ -colourable.
  - q) In  $F_{17}^2$ .  $(v_9 < v_8 < v_2 < v_6 < v_4 < v_5 < v_3 < v_7 < v_1)$  is  $p_{17}$ -greedy, so by Lemma 14,  $F_{17}^2$  is  $p_{17}$ -choosable.

□

**Proof of Lemma 6.**

5)] Suppose for a contradiction that  $v_1$  and  $v_3$  are in  $Z_5$ . Let  $v_5$  (resp.  $v_6$ ) be the neighbour of  $v_1$  (resp.  $v_3$ ) distinct from  $v_2$  and  $v_4$ . By Lemma 5-2),  $v_5 \neq v_6$ .  $(G[S], p_S) = (F_1, p_1)$  and  $G^2[S] \subset F_1^2 \cup \{v_4v_5, v_5v_6, v_6v_4\}$ . So Lemma 24 contradicts Lemma 12.



Suppose for a contradiction that, in  $K$ , a vertex  $v_4$  of  $Y_{1,2,2}$  is adjacent to two vertices of  $Y_{1,2,2}$   $v_2$  and  $v_6$  by 2-edges. According to 5),  $v_2 \neq v_6$ . Let  $v_3$  and  $v_5$  be the 2-neighbours of  $v_4$  common with  $v_2$  and  $v_6$  respectively, and  $v_1$  (resp.  $v_7$ ) be the neighbour of  $v_2$  (resp.  $v_6$ ) not adjacent to  $v_4$ . Set  $S = \{v_1, \dots, v_6, v_7\}$ .

- Assume that  $v_2v_4$  is an edge. For  $i \in \{1, 4, 7\}$ , let  $w_i$  be the neighbour of  $v_i$  not in  $S$ .  $(G[S], p_S) = (F_2, p_2)$  and  $G^2[S] \subset F_2^2 \cup \{v_1v_4, v_4v_7, v_1v_7\}$ . Thus, by Lemmas 24 and 12,  $w_1 = w_7 = v_8$ . Let  $T = S \cup \{v_8\}$ . If  $v_8 \neq w_4$ , then  $(G[T], p_T) = (F_3, p_3)$  and  $G^2[T] \subset F_3^2 \cup \{v_4v_8\}$ . So Lemma 24 contradicts Lemma 12. If not then  $G[T] = G = F_4$ , so  $G$  is 6-choosable, by Lemma 24. This is a contradiction.
- Assume that  $v_1 = v_7$ . Clearly  $(G - S)^2 = G^2 - S$  because every vertex of  $C$  has at most one neighbour in  $G - S$ . Moreover  $C$  has no chord by Lemma 5-2),  $C^2 = G^2[S]$ . Finally  $p_S(v_i) = 4$  if  $i$  is even and  $p_S(v_i) = 3$  otherwise.  $C^2$  is a cycle+triangle graph, thus, by Theorem 21, it is 3-choosable and so  $p_S$ -choosable. This contradicts Lemma 12.
- Assume that  $v_1 \neq v_7$ . For  $i \in \{1, 2, 4, 6, 7\}$ , let  $w_i$  be the neighbour of  $v_i$  not in  $S$ . Let  $W = \{w_1, w_2, w_4, w_6, w_7\}$ .

Suppose first that  $W \cap S \neq \emptyset$ . Since  $G$  is simple,  $w_1 \neq v_2$  and  $w_7 \neq v_6$  and by Lemma 5-3),  $w_1 \neq v_4$  and  $w_7 \neq v_4$ . Moreover, by (5),  $w_1 \neq v_6$  and  $w_7 \neq v_2$ . Then, by symmetry, we only need to consider the cases  $w_2 = v_4$  and  $w_2 = v_6$ .

- Suppose that  $w_2 = v_4$ . Set  $R = \{v_1, v_2, v_3, v_4, v_5\}$ , then  $(G[R], p_R) = (F_2, p_2)$  and  $G^2[R] \subset F_2^2 \cup \{v_1v_5\}$ . Thus Lemma 24 contradicts Lemma 12.
- Suppose that  $w_2 = v_6$ . Then  $(G[S], p_S) = (F_3, p_3)$ . If  $w_1 \neq w_7$  then  $G^2[S] \subset F_3^2 \cup \{v_1v_4\}$  or  $G^2[S] \subset F_3^2 \cup \{v_4v_7\}$ . So Lemma 24 contradicts Lemma 12. Hence we may assume that  $w_1 = w_7 = v_8$ . Let  $T = S \cup \{v_8\}$ . If  $v_8 \neq w_4$ , then  $(G[T], p_T) = (F_4, p_4)$  and  $G^2[T] \subset F_4^2 \cup \{v_4v_8\}$ . So Lemma 24 contradicts Lemma 12. If  $v_8 = w_4$  then  $G = G[T] = F_5$  which, according to Lemma 24, is 6-choosable, a contradiction.

Hence, we may assume that  $W \cap S = \emptyset$ .

Note that by Lemma 5-2),  $w_1 \neq w_2$  and  $w_6 \neq w_7$ .

Suppose  $w_1 = w_4 = v_8$ . Then let  $R = \{v_1, v_2, v_3, v_4, v_5, v_8\}$  and  $w_8$  the neighbour of  $v_8$ . Then  $(G[R], p_R) = (F_6, p_6)$ . Recall that  $w_2 \neq v_6$ . So  $G^2[R] = F_6^2$ . Thus Lemma 24 contradicts Lemma 12.

Therefore, we may assume that  $w_1 \neq w_4$  and, by symmetry,  $w_4 \neq w_7$ .

Suppose  $w_1 = w_7$ . Let  $T = S \cup \{v_8\}$ . Then  $G[T]$  is the cycle  $C_8$  and  $p_T$  is the function  $p$  defined in Lemma 22. So by Lemmas 22 and 12,  $G^2[T] \neq C_8^2$ . It follows that either  $w_2 = w_6$  or  $w_4 = w_8$  with  $w_8$  be the neighbour of  $v_8$  not in  $S$ .

- Suppose  $w_2 = w_6 = v_9$  and  $w_4 = w_8 = v_{10}$ . Set  $W = \{v_1, \dots, v_{10}\}$ . If  $v_9v_{10} \notin E(G)$  then  $(G[W], p_W) = (F_7, p_7)$  and  $G^2[W] \subset F_7^2 \cup \{v_9v_{10}\}$ ; so Lemma 24 contradicts Lemma 12. If not,  $G = G[W] = F_{13}$ , so  $G^2$  is 6-choosable, according to Lemma 24, a contradiction.
- Suppose  $w_2 = w_4 = w_6 = v_9$ . Setting  $U = \{v_1, \dots, v_9\}$ , we have  $G[U] = F_8$  and  $G^2[U] = F_8^2$ . Hence Lemma 24 contradicts Lemma 12.  
By symmetry, we get a contradiction if  $w_2 = w_6 = w_8$ ,  $w_2 = w_4 = w_8$  or  $w_4 = w_6 = w_8$ .
- Suppose  $w_4 = w_8 = v_9$ ,  $w_2 \neq v_9$ ,  $w_6 \neq v_9$  and  $w_2 \neq w_6$ . Setting  $U = \{v_1, \dots, v_9\}$ , we have  $(G[U], p_U) = (F_9, p_9)$  and  $G^2[U] \subset F_9^2 \cup \{v_2v_9\}$  or  $G^2[U] \subset F_9^2 \cup \{v_6v_9\}$ . Hence Lemma 24 contradicts Lemma 12.  
By symmetry, we get a contradiction if  $w_2 = w_6 = v_9$ ,  $w_4 \neq v_9$ ,  $w_8 \neq v_9$  and  $w_4 \neq w_8$ .

Therefore, we may assume that  $w_1 \neq w_7$ .

Suppose that  $w_2 = w_6 = v_8$ . Let  $T = S \cup \{v_8\}$ . Then  $(G[T], p_T) = (F_{10}, p_{10})$ , and  $G^2[T] \subset F_{10}^2 \cup \{v_4v_8\}$ , since  $w_1$ ,  $w_4$  and  $w_7$  are distinct vertices. Hence Lemma 24 contradicts Lemma 12.

Therefore, we may assume that  $w_2 \neq w_6$ .

Suppose that  $w_1 = w_6 = v_8$  and  $w_2 = w_7 = v_9$ . Let  $U = S \cup \{v_8, v_9\}$ . If  $v_8v_9 \notin E(G)$  then  $(G[U], p_U) = (F_{11}, p_{11})$ , and  $G^2[U] \subset F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ . If not, then  $(G[U], p_U) = (F_{14}, p_{14})$ , and  $G^2[U] = F_{14}^2$ . In both cases, Lemma 24 contradicts Lemma 12.

Therefore, we may assume that  $w_1 \neq w_6$  or  $w_2 \neq w_7$ . By symmetry, we may assume that  $w_2 \neq w_7$ .

Suppose  $w_1 = w_6 = v_8$ . Let  $T = S \cup \{v_8\}$  and let  $w_8$  be the neighbour of  $v_8$  not in  $S$ . Then  $(G[T], p_T) = (F_{12}, p_{12})$  and  $G^2[T] \subset F_{12}^2 \cup \{v_4v_8\}$ . Hence Lemma 24 contradicts Lemma 12.

Therefore, we may assume that  $w_1 \neq w_6$ .

Hence we have  $w_1 \neq w_4, w_6, w_7$ ,  $w_2 \neq w_6, w_7$  and  $w_4 \neq w_7$  then  $G[S]^2 = G^2[S]$ . Thus Proposition 23 contradicts Lemma 12.

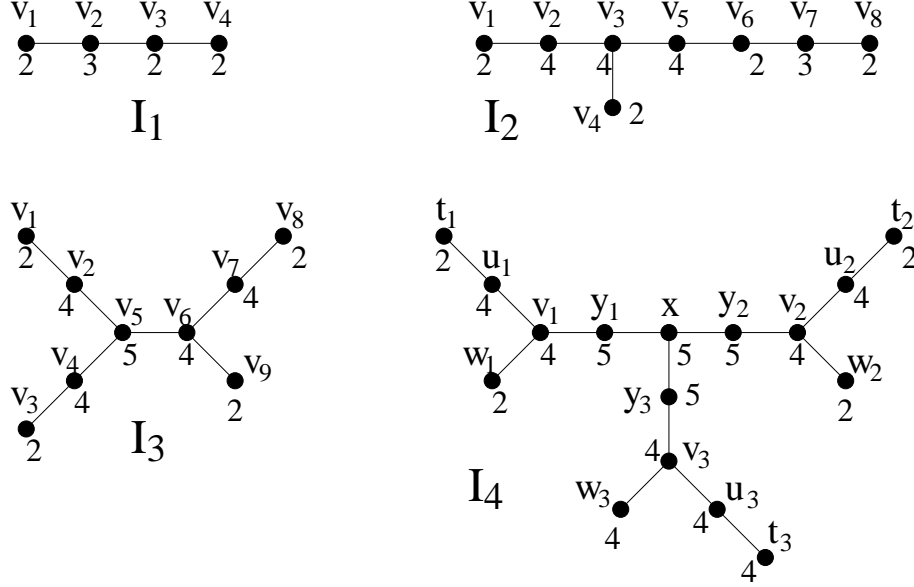
□

### 3.3 Proof of Lemma 11

**Definition 25** For  $1 \leq j \leq 4$ , let  $I_j$  and  $q_j$  be the graphs and function depicted Figure 4

**Lemma 26** For  $1 \leq j \leq 4$ ,  $I_j^2$  is  $q_j$ -choosable.

**Proof.**

Figure 4: The graphs  $I_j$  and functions  $q_j$ ,  $1 \leq j \leq 4$ 

- Let  $L$  be a  $q_1$ -list-assignment of  $I_1^2$ . ( $v_4 < v_3 < v_1 < v_2$ ) and ( $v_1 < v_3 < v_4 < v_2$ ) are  $q_1$ -nice, so by Lemma 15, we may assume that  $L(v_1) \cup L(v_4) \subset L(v_2)$ . Hence  $L(v_1) \cap L(v_4) \neq \emptyset$ . But ( $v_4 < v_1 < v_3 < v_2$ ) is  $q_1$ -good. Thus by Lemma 16,  $I_1^2$  is  $L$ -colourable.
- Let  $L$  be a  $q_2$ -list-assignment of  $I_2^2$ .  
Suppose first that  $L(v_3) \not\subset L(v_1) \cup L(v_6)$ . Then choose  $c(v_3)$  in  $L(v_3) \setminus (L(v_1) \cup L(v_6))$  and  $c(v_4) \in L(v_4) \setminus \{c(v_3)\}$ . Since  $I_1^2$  is  $q_1$ -choosable, one can extend  $c$  to  $\{v_5, v_6, v_7, v_8\}$ . Then one can find  $c(v_2) \in L(v_2) \setminus \{c(v_3), c(v_4), c(v_5)\}$  and  $c(v_1) \in L(v_1) \setminus \{c(v_2), c(v_3)\} = L(v_1) \setminus \{c(v_2)\}$ . So, we may assume that  $L(v_3) \subset L(v_1) \cup L(v_6)$ , so  $L(v_3) = L(v_1) \cup L(v_6)$  and  $L(v_1) \cap L(v_6) = \emptyset$ .  
Now colour  $v_3$  and  $v_6$  the same colour  $c_6 \in L(v_6)$ . Then proceed greedily according to ( $v_4 < v_8 < v_7 < v_5 < v_2 < v_1$ ). It is possible since  $c_6 \notin L(v_1)$ .
- Let  $L$  be a  $q_3$ -list-assignment of  $I_3^2$ . Assign to  $v_5$  a colour  $c_5$  in  $L(v_5) \setminus (L(v_1) \cup L(v_9))$  and to  $v_6$  a colour in  $L(v_6) \setminus (L(v_8) \cup \{c_5\})$ . Then colour the remaining vertices greedily according to ( $v_3 < v_4 < v_2 < v_1 < v_1 < v_9 < v_7 < v_8$ ) to get an  $L$ -colouring of  $I_3^2$ .
- Let  $L$  be  $q_4$ -list-assignment of  $I_4^2$ . Pick  $c(y_1)$  in  $L(y_1) \setminus L(w_1)$ ,  $c(y_2)$  in  $L(y_2) \setminus (L(w_2) \cup \{c(v_1)\})$ ,  $c(y_3)$  in  $L(y_3) \setminus (L(w_3) \cup \{c(v_1), c(v_2)\})$  and  $c(x)$  in  $L(x) \setminus \{c(v_1), c(v_2), c(v_3)\}$ . Since  $I_1^2$  is  $q_1$ -choosable, one can extend  $c$  to a colouring of  $I_4^2$ .

□

**Proof of Lemma 6.**

- 3) Suppose that a vertex  $v_3$  of  $Y_{2,2,3}$  and  $v_6$  of  $Y_{1,2,3}$  are adjacent via a 2-edge in  $K$ . Then the subgraph of  $G$  induced by  $v_3$ ,  $v_6$  and the 2-vertices of their incident threads is  $I_2$ . Since  $G$  has girth at least 13, then  $G^2[I_2] = I_2^2$  and  $(G - V(I_2))^2 = G^2 - V(I_2)$ , so Lemma 26 contradicts Lemma 12.
- 4) Suppose that a vertex  $v_5$  of  $Y_{1,3,3}$  and  $v_6$  of  $Y_{1,2,3}$  are adjacent via a 1-edge in  $K$ . Then the subgraph of  $G$  induced by  $v_5$ ,  $v_6$  and the 2-vertices of their incident threads is  $I_3$ . Then Lemma 26 contradicts Lemma 12.
- 5) Suppose that a vertex  $x$  of  $Y_{2,2,2}$  is adjacent to three vertices  $v_1$ ,  $v_2$  and  $v_3$  of  $Y_{2,2,3}$  in  $K$ . Then the subgraph of  $G$  induced by  $x$ ,  $v_1$ ,  $v_2$ ,  $v_3$  and the 2-vertices of their incident threads is  $I_4$ . Then Lemma 26 contradicts Lemma 12.

□

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